Chaotic dynamics of cold atoms in far-off-resonant donut beams

X. M. Liu and G. J. Milburn
The Center for Laser Sciences, Department of Physics, The University of Queensland, St. Lucia, Brisbane, Queensland 4072, Australia
(Received 9 September 1998)

We describe the classical two-dimensional nonlinear dynamics of cold atoms in far-off-resonant donut beams. We show that chaotic dynamics exists there for charge greater than unity, when the intensity of the beam is periodically modulated. The two-dimensional distributions of atoms in the (x,y) plane for charge 2 are simulated. We show that the atoms will accumulate on several ring regions when the system enters a regime of global chaos. [S1063-651X(99)03903-3]

PACS number(s): 05.45.−a

I. INTRODUCTION

The Laguerre-Gaussian beams (donut beams) carry orbital angular momentum associated with a helical surface of constant phase and have recently been the subject of considerable theoretical and experimental study [1–7]. Many of these studies concern the transfer of orbital angular momentum of the beam to small particles and atoms [1]. With the development of laser cooling and trapping of neutral atoms, various schemes to slow the atomic motion, and increase the atomic intensity, have been proposed and some demonstrated. In a far-off-resonant laser beam it is possible to observe the spatial variation of the optical dipole force on slow atoms. In this case the atom experiences an effective mechanical potential proportional to the intensity of the beam. Such forces can be used to trap cold atoms and recently Gaussian-Laguerre modes were used to trap atoms [6].

The degree of nonlinearity of the optical potential in a donut beam is characterized by the ''charge'' of the donut. The charge is an integer characterizing the phase singularity of the beam on axis. It also determines the variation of the intensity off-axis in the transverse plane. In the case of unit charge the intensity close to the axis varies quadratically with the radial distance in the transverse plane. For a charge greater than unity the variation of intensity is a quartic, or higher, power of the radical distance and thus the resulting motion near the beam axis is nonlinear. In this case the frequency of bound oscillatory motion near the axis depends on the energy of the atom. As the energy increases, the frequency decreases, eventually falling to zero as the unstable fixed point at the potential maximum is reached. If such a nonlinear system is driven by a periodic modulation of the potential, regions of chaotic orbits can result [11]. In this paper we will consider the motion of cold atoms in the intensity modulated Laguerre-Gaussian beam and discuss the classical two-dimensional chaotic dynamics for charge $l=2$.

II. LAGUERRE-GUASSIAN (LG) BEAM AND DONUT BEAM

The field of a linearly polarized LG beam in cylindrical coordinates, (r,ϕ,z), is [3]

$$\mathbf{E}(\mathbf{r})=\mathbf{e}_x\frac{C}{1+z^2/Z_R^2}\left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|}L_q^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\times\exp(-r^2/w^2(z))\exp\left(\frac{ikr^2z}{2(z^2+z_R^2)}\right)\times\exp[il\phi]\exp[i(2q+l+1)\tan^{-1}(z/z_R)]\exp(ikz),\tag{1}$$

where $z_R=\pi w_0^2/\lambda, w^2(z)=\lambda(z^2+z_R^2)/\pi z_R=n-m$ and $q=\min(n,m)$ for the $E_{m,n}^{\mathrm{LG}}$ mode.

We are concerned here with the donut beam, because it is the easiest way to form an optical trap in the transverse plane. In this case $q=0$ and therefore $L_0^{|l|}()=0$. In reality, Eq. (1) can be greatly simplified because in general $z\ll z_R$ and $w(z)\simeq w_0$.

The new expression of the electric field of the donut beam is

$$\mathbf{E}(\mathbf{r})=\mathbf{e}_xC\left(\frac{r\sqrt{2}}{w_0}\right)^{|l|}\exp(-r^2/w_0^2)\exp[il\phi]\exp(ikz),\tag{2}$$

where l is the topological charge of singularity, which can be positive or negative. Each photon in a linearly polarized donut beam carries $L_z=l\hbar$ orbital angular momentum. The intensity distribution of the beam when $q=0$ is given by [6]

$$I(\mathbf{r})=W\frac{2^{|l|+1}r^{2|l|}}{\pi|l|!\,w_0^{2(|l|+1)}}\exp[-2r^2/w_0^2],\tag{3}$$

where W is the power of a laser.

III. CHAOTIC DYNAMICS OF COLD ATOMS IN FAR-OFF-RESONANCE DONUT BEAMS

Spontaneous emission for cold atoms in far-off-resonant donut beams can be ignored. The interaction with the light field is then almost perfectly Hamiltonian. The effective optical potential for a two-level atom [8] has the form

$$U(\mathbf{r})=\frac{\hbar\Delta}{2}\ln(1+p),\tag{4}$$

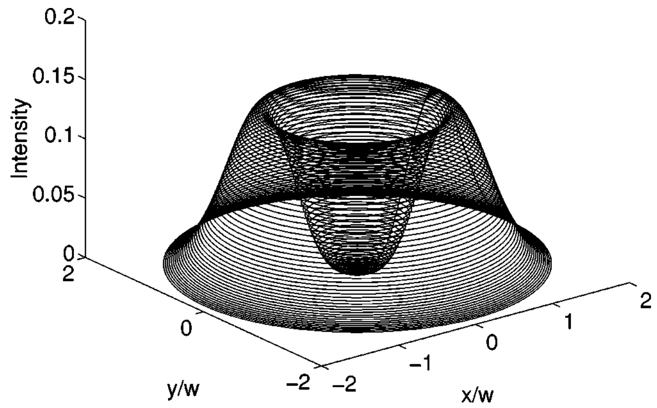

FIG. 1. The relative intensity of a donut beam for charge $l=2$, where w is the beam waist.

where Δ is the detuning and $p=\Omega^2/2/\Delta^2+\Gamma^2/4$ is a saturation parameter, with the Rabi frequency Ω. For far-off-resonant donut beams, $p\ll 1$, and thus

$$U(\mathbf{r})=\frac{\hbar\Omega(\mathbf{r})^2}{4\Delta}. \tag{5}$$

Taking into account the special variation of the Rabi frequency, the classical Hamiltonian in the (x,y) plane for the system is

$$H_0=\frac{p_x^2+p_y^2}{2M}+K(l)(x^2+y^2)^l\exp\left(-2\frac{x^2+y^2}{w^2}\right), \tag{6}$$

where $K(l)=(\hbar\Omega_0^2/2\Delta)(2^l/l!w_0^{2l})$ and $\Omega_0=\Gamma\sqrt{W/2\pi w_0^2 I_s}$; I_s is the saturation intensity. In the case of rubidium, $I_s=2\ \text{mW/cm}^2$.

When the Rabi frequency is modulated, Ω_0 becomes time dependent, $\Omega_0\sqrt{1+\epsilon\cos(\omega t)}$, and the dynamics of the atom can be chaotic for certain initial conditions. The effective optical potential depends on the charge, l, of the donut. The larger the value of l, the more anharmonic is the potential on axis. For $l=1$, the potential is approximately quadratic near the axis, and the unperturbed motion of an atom is harmonic. In that case no chaotic orbits will occur when the potential is modulated in time. If, however, the atom has a larger kinetic energy so that it is moving in a region far from the axis and can explore the nonlinear parts of the potential, we expect the dynamics to became chaotic. When $l>1$, the dynamics is nonlinear over the whole potential range and the amplitude of the modulation is easier to control as the potential is wider and flatter.

For simplicity we discuss the $l=2$ case (Fig. 1) and we take rubidium as a particular example to set the parameters for our simulation. The parameters for rubidium are linewidth $\Gamma/2\pi=6$ MHz, mass $M=85m_p$, the beam waist of laser $w_0=140\ \mu$m, laser power $W=600\ \text{mW/cm}^2$, and the detuning $\Delta/2\pi=6$ GHz. We now define dimensionless parameters $(x,y)=(\tilde{x},\tilde{y})=(x/w,y/w),(\widetilde{p_x},\widetilde{p_y})=(p_x/P_D, p_y/P_D)$, and $\tilde{H}=H/2E_D$ and $\tilde{t}=t/(w/MP_D)$, where $E_D=P_D^2/2M$ is the Doppler limit energy and P_D is the momentum, respectively. Omitting the tildes and defining $\xi=\hbar\Omega^2/2\Delta E_D$, for charge $l=2$, the Hamiltonian can be rewritten as

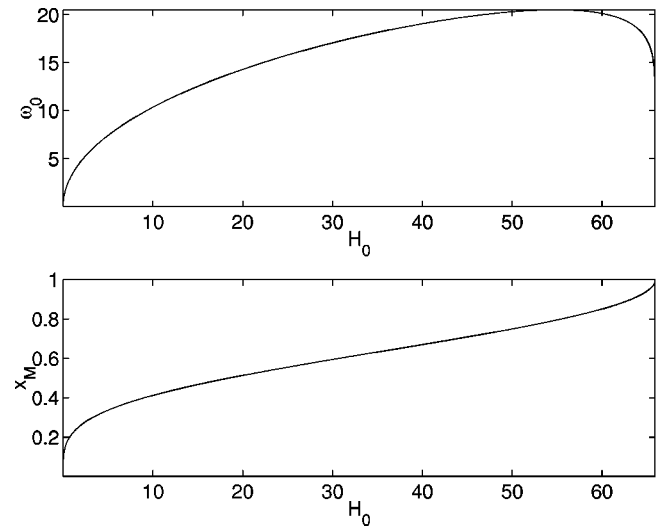

FIG. 2. The relations between frequency of motion and Hamiltonian $\omega_0\sim H_0$ and $x_M\sim H_0$.

$$H(t)=\frac{p_x^2+p_y^2}{2}+\xi(l)(x^2+y^2)^2e^{-2(x^2+y^2)}(1+\epsilon\cos\omega t), \tag{7}$$

where $\xi\approx 0.887$. Using Hamilton's equations we find that the motion in the transverse plane without modulation ($\epsilon=0$) is described by the equations

$$\dot{p}_x=-4\xi xr^2(1-r^2)e^{-2r^2}, \tag{8}$$

$$\dot{p}_y=-4\xi yr^2(1-r^2)e^{-2r^2}, \tag{9}$$

$$\dot{x}=p_x, \tag{10}$$

$$\dot{y}=p_y, \tag{11}$$

where $r^2=x^2+y^2$. Clearly there are two fixed points: one stable fixed point on axis ($r=0$), and one unstable fixed point at the intensity maximum ($r=1$).

The choice of modulation frequency ω depends on the frequency of unperturbed periodic motion. For simplicity we assume $y=0$ and $p_y=0$, so the expression for H simplifies to a one-dimensional Hamiltonian system. The period of motion for the unperturbed Hamiltonian H_0 is [9]

$$T=\oint\frac{dx}{\partial H_0/\partial p_x}=2\int_{-x_M}^{x_M}\frac{dx}{\sqrt{2[H_0-\xi x^2\exp(-2x^2)]}}, \tag{12}$$

where x_M is determined by $H_0=\xi x_M^2\exp(-2x_M^2)$. Therefore

$$\omega_0=\frac{\pi}{\int_{-x_M}^{x_M}\{2[H_0-\xi x^2\exp(-2x^2)]\}^{-1/2}dx}. \tag{13}$$

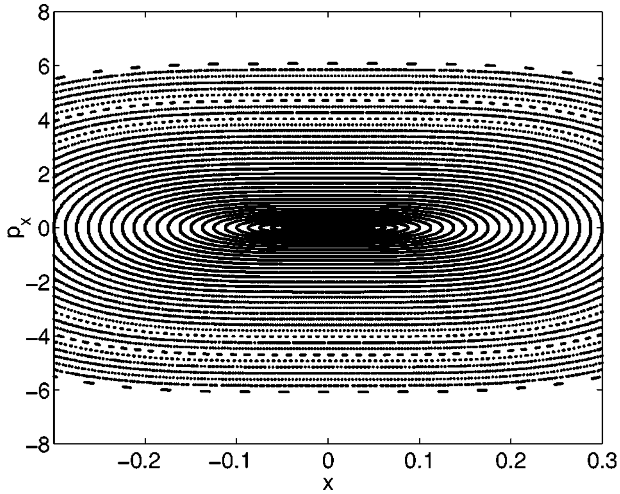

FIG. 3. Stroboscopic portrait of the system with $\epsilon=0$, $p_x(0)=0$, $p_y=0$, and $y=0$. The maximum strobe number is 500.

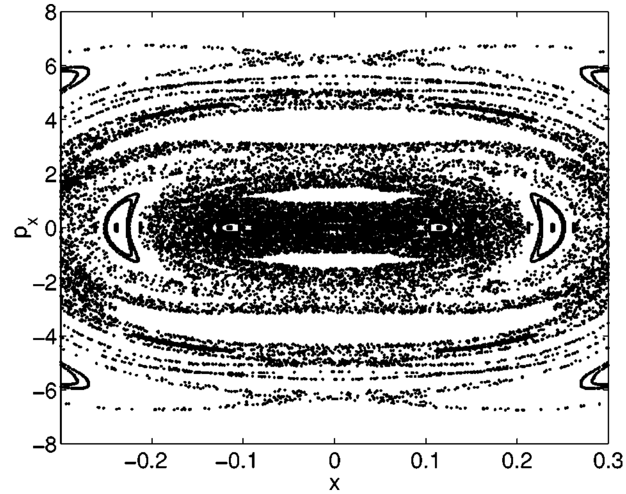

FIG. 5. Stroboscopic portrait of the system with $\epsilon=0.7$, $p_x(0)=0$, $p_y=0$, and $y=0$. The maximum strobe number is 500.

The graph of ω versus H_0 and x_M versus H_0 is shown in Fig. 2. We can select the modulation frequency ω to control the position of the fixed points. Here we set the dimensionless modulation frequency, $\omega=4.34$, which corresponds to 3.67 KHz.

We use a symplectic integration routine [10,11] to solve the equations of motion so as to preserve the Poisson bracket relation $\{x(t),p_x(t)\}=1$, and thus maintain the Hamiltonian character of the motion. In Figs. 3–5 we plot the stroboscopic portrait of the system at times $t=(2\pi/\omega)s$, where s is an integer referred to as the strobe number. From Figs. 3–5 we can see that regions of chaotic motion will arise as ϵ is increased, together with some regular regions. A broad initial phase space distribution of atoms will enable some atoms to become trapped in these stable regions.

Laser cooling and trapping techniques have the ability to cool the atom to very low velocities and trap them with well localized momentum, however the position distribution is not so well localized. Therefore the appropriate description of the initial conditions is in terms of a probability density on phase space (x,y,p_x,p_y). We define a classical state to be a probability measure on phase space of the form $Q(x,y,p_x,p_y)dxdydp_xdp_y$. The probability density satisfies the Liouville equation

$$\frac{\partial Q}{\partial t}=\{H,Q\}_{q_i,p_i}, \tag{14}$$

where $\{,\}_{q_i,p_i}$ is the Poisson bracket. This equation can be solved by the method of characteristics. To simulate an experiment, we assume atoms are initially uniformly distributed on the $|x|<C$ and $|y|<C$ region, where C is a constant chosen to ensure the major fixed points are included. The momentum distributions for p_x and p_y are assumed to be Gaussian distributions. Therefore

$$Q_0(x,y,p_x,p_y)=Q_0(x)Q_0(y)Q_0(p_x)Q_0(p_y), \tag{15}$$

where

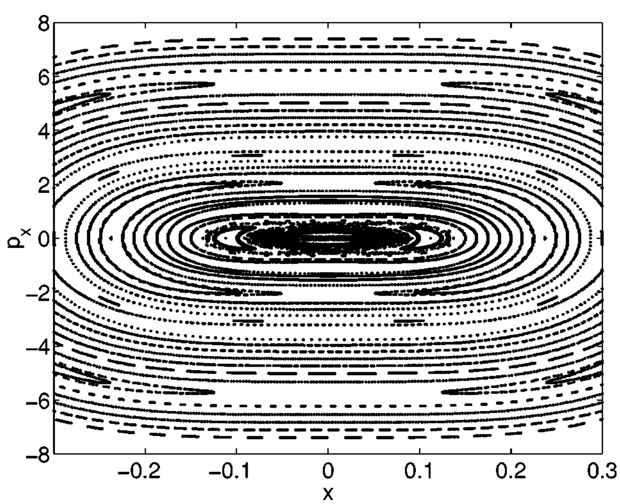

FIG. 4. Stroboscopic portrait of the system with $\epsilon=0.5$, $p_x(0)=0$, $p_y=0$, and $y=0$. The maximum strobe number is 500.

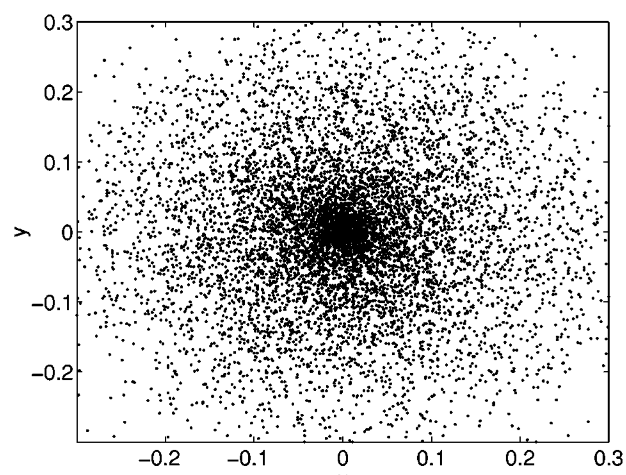

FIG. 6. The atomic distribution in (x,y) plane at the strobe number 50 for $\epsilon=0$. The 10 000 points were taken in phase space. The atoms were initially distributed on $|x|\leqslant 0.3$ and $|y|\leqslant 0.3$ region. The momenta of p_x,p_y are Gaussian distributions and $\sigma_{p_x}=\sigma_{p_y}=0.05$.

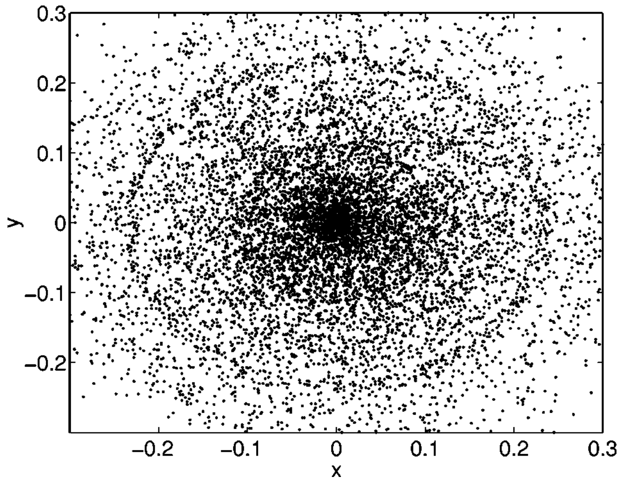

FIG. 7. The atomic distribution in (x,y) plane at the strobe number 50 for $\epsilon=0.5$. The 10 000 points were taken in phase space. The atoms were initially distributed on $|x|\leq 0.3$ and $|y|\leq 0.3$ region. The momenta of p_x, p_y are Gaussian distributions and $\sigma_{p_x}=\sigma_{p_y}=0.05$.

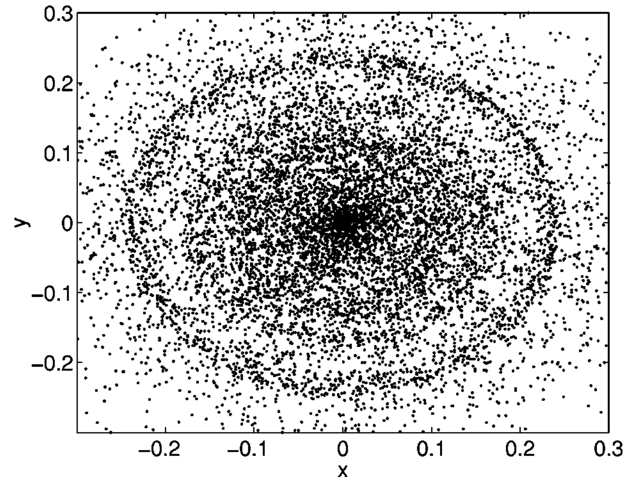

FIG. 8. The atomic distribution in (x,y) plane at the strobe number 50 for $\epsilon=0.7$. The 10 000 points were taken in phase space. The atoms were initially distributed on $|x|\leq 0.3$ and $|y|\leq 0.3$ region. The momenta of p_x, p_y are Gaussian distributions and $\sigma_{p_x}=\sigma_{p_y}=0.05$.

$$Q_0(p_i)=\frac{1}{2\pi\sigma_{p_i}}\exp\{-[p_x-p_x(0)]^2/2\sigma_{p_i}\}. \tag{16}$$

The variances of p_x and p_y are related to the temperature T_i,

$$\sigma_{p_i}=Mk_BT_i/P_D^2. \tag{17}$$

Two-dimensional symplectic integrators [10] are used to preserve the Poisson bracket relations during computation,

$$\{q_i, p_j\}=\delta_{ij}. \tag{18}$$

In Fig. 6 we show the case for no modulation. In this case atoms will accumulate around the fixed point $x=y=0$ (see Fig 3). When the modulation is added, the atoms will diffuse in regions of chaotic motion but some will accumulate around several rings corresponding to fixed points at nonzero radius. With the increase of modulation amplitude, more atoms accumulate around rings and fewer atoms around $x=y=0$ (Figs. 7 and 8). In our simulations the variances σ_{p_x} and σ_{p_y} are taken to be 0.05, which corresponds to temperatures at approximately the recoil cooling limit. As the variance decreases (lower temperature) the radial rings become clearer.

IV. CONCLUSION AND DISCUSSION

We have shown that an atom moving in an intensity modulated far-off-resonance donut beam can exhibit chaotic dynamics in the transverse plane. For atomic momenta p_x, p_y with Gaussian distributions, some atoms will become trapped in rings corresponding to radial fixed points of the modulated system. If at some moment the optical potential is withdrawn, atoms will expand freely and the spatial structure of the rings will persist because the momentum is symmetric in (x,y). Therefore the two dimensional radial atomic distribution in (x,y) can be detected using standard time of flight techniques.

ACKNOWLEDGMENT

Dr. S. Dyrting at Hong Kong University of Science and Technology gave one of the authors (X.M.L.) helpful advice about two-dimensional symplectic integrators.

[1] H. He, M.E.J. Friese, N.R. Heckenberg, and H. Rubinsztein-Dunlop, Phys. Rev. Lett. **75**, 826 (1995).

[2] Iwo Bialynicki-Birula, and Zofia Bialynicki-Birula, Phys. Rev. Lett. **78**, 2539 (1997).

[3] W.L. Power, L. Allen, M. Babiker, and V.E. Lembessis, Phys. Rev. A **52**, 479 (1995).

[4] M. Babiker, W.L. Power, and L. Allen, Phys. Rev. Lett. **73**, 1239 (1994).

[5] L. Allen, M.W. Beijersbergen, R.J.C. Spreeuw, and J.P. Woerdman, Phys. Rev. A **45**, 8185 (1992).

[6] Takahiro Kuga, Yoshio Torii, Noritsugu Shiokawa, and Takuya Hirano, Y. Shimizu and H. Sasada, Phys. Rev. Lett. **78**, 4713 (1997).

[7] Wenyu Chen, S. Dyrting, and G.J. Milburn, Aust. J. Phys. **49**, 777 (1996).

[8] C. Cohen-Tannoudji, in *Fundamental Systems in Quantum Optics*, Proceedings of the Les Houches Summer School (North-Holland, Amsterdam, 1992).

[9] A.J. Lichtenberg and M.A. Lieberman, *Regular and Stochastic Motion* (Springer-Verlag, New York, 1983).

[10] Etienne Forest and Martin Berz, *Canonical Integration and Analysis of Periodic Maps Using Non-standard Analysis and Lie Methods*, in Lie Methods in Optics II, edited by Kurt Bernardo Wolf (Springer-Verlag, Berlin, 1989), p. 47.

[11] S. Dyrting, Ph.D. thesis, Department of Physics, University of Queensland, 1995 (unpublished).